## THE SEVENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Seventh Annual Meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Friday, December 28, 1900. An important feature of the occasion was the election of officers and other members of the Council, at which recently adopted amendments of the Constitution went into effect. The term of office of the President, who must have served previously, but with a year's interval, as Vice-President, is now increased to two years. Both Presidents and Vice-Presidents are made ineligible for immediate reëlection. Four members of the Council, instead of three, are now elected annually. This provision and the permanent membership of the ex-Presidents increases the number of seats in the Council to twenty-four at present The growth of the Society has been such that the numerical basis of representation of the Council is almost precisely the same as that of six years ago, when the Society was reorganized as a national body.

At the close of the century a brief recapitulation of the advance of the Society during these six years is not inappropriate. The record reveals in the past a happy and substantial augury for the future. The membership has grown from 244 in September, 1894. to 357 on January 1, 1901. In 1894 the number of papers read at the meetings was 24 ; in 1900 it was 112. This expansion in numbers and output has had for one valuable result the creation, in 1898, of the Chicago Section of the Society, an event which has proved not only a great benefit to many members to whom the New York meetings are practically inaccessible, but a source of strength to the Society as a whole. The demand for improved facilities for publication, occasioned by the great increase in productive activity of the members, has had for its outcome the founding of the *Transactions*, which has now successfully completed its first year of existence as the official organ of the Society for the publication of important original papers read before it. Meanwhile the BULLETIN has been considerably increased in size, although confining itself more strictly than before to the historical and critical field for which it was originally designed. A gratifying recognition of the usefulness and efficiency of the Society is evidenced by the liberal financial coöperation of ten leading universities of the country in the publication of the

*Transactions*. Since 1894 summer meetings, of great interest and largely attended, have been held, usually in connection with large and general scientific gatherings. On two occasions the summer meeting has been supplemented by colloquia, or special courses of lectures summarizing important fields of recent investigation. The attendance at the meetings throughout the year has constantly increased. In 1900 the total attendance was 209, and the number of members attending at least one meeting during the year was 110. The administration of business has been wisely left from the beginning in the hands of the Council, the time of the meetings being thus economized for purely scientific purposes. The finances of the Society are in a satisfactory condition, and the extra expense of publishing the *Transactions* has been met without a reduction of the small surplus which has slowly accumulated in the course of years. Over one-third of this surplus consists, however, of life membership payments, which have been set apart as a separate invested fund to be devoted to such purposes as may hereafter be approved. The Society is accumulating from the exchanges of the BULLETIN and the *Transactions* and from numerous gifts a considerable library of mathematical journals and works. It is to be hoped that this collection may be bound and made accessible to the members generally within a short time. The Society, however, has at present no funds available for this purpose.

The attendance at the annual meeting numbered about forty persons, including the following thirty-three members of the Society :—

Dr. C. L. Bouton, Professor E. W. Brown, Professor F. N. Cole, Dr. W. S. Dennett, Professor T. S. Fiske, Mr. W. B. Ford, Mr. A. S. Gale, Professor Harris Hancock, Dr. G. W. Hill, Dr. A. A. Himowich, Dr. J. I. Hutchinson, Mr. S. A. Joffe, Dr. Edward Kasner, Mr. C. J. Keyser, Professor Pomeroy Ladue, Dr. Emory McClintock, Dr. James Maclay, Professor Frank Morley, Dr. D. A. Murray, Professor W. F. Osgood, Mr. J. C. Pfister, Professor M. I. Pupin, Professor J. K. Rees, Professor C. A. Scott, Professor P. F. Smith, Dr. Virgil Snyder, Dr. W. M. Strong, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Professor A. G. Webster, Professor L. A. Wait, Mr. E. B. Wilson, Professor F. S. Woods.

The meeting extended as usual through a morning and an afternoon session. Vice-President Thomas S. Fiske occupied the chair. The Council announced the election of the following persons to membership in the Society :—Dr. G. N.

Bauer, University of Minnesota, Minneapolis, Minn.; Dr. J. R. Benton, Princeton University, Princeton, N. J.; Dr. H. F. Blichfeldt, Leland Stanford University, Palo Alto, Cal.; Dr. G. A. Bliss, University of Minnesota, Minneapolis, Minn.; Professor Joseph Bowden, Adelphi College, Brooklyn, N. Y.; Professor D. F. Campbell, Armour Institute, Chicago, Ill.; Dr. J. E. Clarke, Gilbert School, Winsted, Conn.; Dr. Euplio Conoscente, New York, N. Y.; Mr. Arthur Crathorne, University of Wisconsin, Madison, Wis.; Mr. H. W. Curjel, Birkdale, Lancashire, Eng.; Professor L. M. Defoe, University of Missouri, Columbia, Mo.; Mr. B. S. Easton, University of Pennsylvania, Philadelphia, Pa.; Dr. L. P. Eisenhart, Princeton University, Princeton, N. J.; Mr. U. S. Hanna, University of Indiana, Bloomington, Ind.; Mr. L. I. Hewes, Yale University, New Haven, Conn.; Mr. A. M. Kenyon, Purdue University, Lafayette, Ind.; Professor C. N. Little, Leland Stanford University, Palo Alto, Cal.; Mr. E. L. Milne, University of Illinois, Champaign, Ill.; Mr. H. B. Mitchell, Columbia University, New York, N. Y.; Dr. Asutosh Mukhopâdyây, Calcutta, India; Mr. W. S. Nichols, New York, N. Y.; Professor J. M. Page, University of Virginia, Charlottesville, Va.; Dr. T. H. Taliaferro, Pennsylvania State College, State College, Pa.; Miss R. H. Vivian, University of Pennsylvania, Philadelphia, Pa., Mr. N. R. Wilson, Royal Military Academy, Kingston, Canada. Six applications for membership in the Society were reported. Reports were received from the Secretary, Treasurer, Librarian, and the Auditing Committee. These reports have been printed in the Annual Register, just issued.

At the annual election the following officers and members of the Council were chosen:

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| *President*, | Professor ELIAKIM H. MOORE. |
| *Vice-Presidents*, | Professor T. S. FISKE,<br>Professor H. S. WHITE. |
| *Secretary*, | Professor F. N. COLE. |
| *Treasurer*, | Dr. W. S. DENNETT. |
| *Librarian*, | Professor POMEROY LADUE. |

*Committee of Publication*,

Professor F. N. COLE,
Professor ALEXANDER ZIWET,
Professor FRANK MORLEY.

*Members of the Council to serve until December, 1903,*

Professor E. W. BROWN,
Professor H. B. FINE,
Professor T. F. HOLGATE,
Professor W. F. OSGOOD.

*Member of the Council to serve until December, 1902,*

Professor E. W. HYDE.

The following papers were presented at this meeting:

(1) Dr. VIRGIL SNYDER: "On some plane curves having factorable parallels."

(2) Professor E. D. ROE: "On a formula of interpolation."

(3) Mr. W. B. FORD: "Dini's method of showing the convergence of Fourier's series and of other allied developments."

(4) Dr. EMORY MCCLINTOCK: "A simplified solution of the cubic."

(5) Professor W. F. OSGOOD: "On the existence of a minimum of the integral

$$\int_{x_0}^{x_1} F(x, y, y')dx$$

when $x_0$ and $x_1$ are conjugate points, and the geodetics on an ellipsoid of revolution;" a revision of a theorem of Kneser.

(6) Mr. C. J. KEYSER: "Theorems concerning positive definitions of finite assemblage and infinite assemblage."

(7) Professor M. I. PUPIN: "Wave propagation over bridged wave conductors."

(8) Professor F. MORLEY: "On a point in Sylvester's theory of canonical forms."

(9) Professor HARRIS HANCOCK: "On primary prime functions in several variables and a generalization of an important theorem of Dedekind."

(10) Dr. J. I. HUTCHINSON: "On some birational transformations of a Kummer surface into itself."

(11) Miss R. G. WOOD: "The collineations of space which transform a non degenerate quadric surface into itself."

(12) Professor H. E. SLAUGHT: "The complete form system of invariants of the group of 120 quadratic Cremona transformations of the plane."

(13) Dr. JAMES MACLAY: "Some geometrical theorems conected with a class of differential equations derived from Poisson's equation

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial x^2} - \frac{mz}{x^2}\right).''$$

(14) Mr. J. K. WHITTEMORE: "Cones of the second degree osculating developable surfaces."

(15) Professor E. O. LOVETT: "The types of line-sphere transformation."

(16) Professor E. O. LOVETT: "Note on differential geometry of $n$-dimensional space."

(17) Dr. L. P. EISENHART: "A demonstration of the impossibility of a triply asymptotic system of surfaces."

(18) Professor MAXIME BÔCHER: "Some cases in which the identical vanishing of the Wronskian is a sufficient condition for linear dependence."

In the absence of the authors, Mr. Whittemore's paper was read by Professor Osgood, and the papers of Professor Roe, Professor Slaught, Professor Lovett, Dr. Eisenhart, and Professor Bôcher were read by title. Dr. Eisenhart's paper appeared in the January number of the BULLETIN. The papers of Mr. Ford, Mr. Keyser, and Dr. Hutchinson are included in the present number. Abstracts of the other papers are given below.

Dr. Snyder's paper is summarized as follows: Taking a scroll contained in a linear congruence and of the type having factorable asymptotic lines, one can transform the directrices into two anti-parallel planes in sphere space, and the generators into spheres which envelope a tubular surface. The lines of curvature are cut from this surface by a series of parallel planes. These sections are projections of parallels to the locus of centers of the surface, and break up into two factors. The same problem is then solved, starting from a surface of revolution, without using line geometry.

Professor Roe extends the formula

$$f(x+y) = (1+\Delta)^y f(x),$$

for a positive integral $y$, to all real values of $y$ within the limits of convergency and for the simplest expansion of the symbol $(1+\Delta)^y$, by the immediate and direct relation of two series of the same function to each other, and independently of any Taylor series. Both symbolic and non-symbolic proofs are given. The following symbolic proof is given for the extension to a positive fractional commensurable $y$. Two series of the same function are taken

$$f(x+0)=u_0,\ f(x+1)=u_1,\ \cdots,\ f(x+r)=u_r,\ \cdots$$

with differences $\Delta u$'s,

$$f(x+0)=v_0,\ f\left(x+\frac{1}{\lambda}\right)=v_1,\ \cdots,\ f\left(x+\frac{\nu}{\lambda}\right)=v_\nu\ \cdots$$

with differences $\delta v$'s.

$\nu$ and $\lambda$ are integers. It is shown that $v_\nu=(1+\delta)^\nu v_0$; then it is easily found that $\Delta^r u_0=[(1+\delta)^\lambda-1]^r v_0$; hence, with the confined meaning of $(1+\Delta)^{\frac{\nu}{\lambda}}$, that

$$(1+\Delta)^{\nu'}u_0=u_0+\binom{\nu'}{1}\Delta u_0+\binom{\nu'}{2}\Delta^2 u_0+\cdots+\binom{\nu'}{r}\Delta^r u_0+\cdots$$

$$=v_0+\binom{\nu'}{1}[(1+\delta)^\lambda-1]v_0+\binom{\nu'}{2}[(1+\delta)^\lambda-1]^2 v_0$$

$$+\cdots+\binom{\nu'}{r}[(1+\delta)^\lambda-1]^r v_0+\cdots$$

(where $\nu'=\nu/\lambda$)

$$=[1+(1+\delta)^\lambda-1]^{\nu'}v_0=(1+\delta)^{\nu'\lambda}v_0=(1+\delta)^\nu v_0$$

$$=v_\nu=f\left(x+\frac{\nu}{\lambda}\right),\ \text{or}\ f\left(x+\frac{\nu}{\lambda}\right)=(1+\Delta)^{\frac{\nu}{\lambda}}f(x).$$

Extension is then made to an integral negative $y$, next to a negative fractional commensurable $y$, and lastly by limits to an incommensurable value of $y$, thus securing the extension to all real values of $y$ for a convergent expansion.

Dr. McClintock's "simplified solution of the cubic" is essentially a rearrangement, in a strikingly simple form, of Bézout's "earlier solution," with certain novel features. It will appear in the *Annals of Mathematics*.

In a paper entitled "Zur Variationsrechnung" (*Mathematische Annalen*, volume 50 (1897), p. 50), Kneser enunciates the theorem that the integral

$$I=\int_{x_0}^{x_1}F(x,\ y,\ y')dx$$

ceases to be a minimum, not only when the interval $(x_0, x_1)$ contains in its interior a point $x'$ conjugate to $x_0$, but when $x_1$ coincides with $x'$. This theorem is true in general, but

not in all cases, and it is the object of Professor Osgood's paper to point out a general class of cases for which it is not true. The geodetics on an ellipsoid of revolution are a case in point. The paper will appear in the *Transactions*.

In Professor Pupin's paper the wave conductor is the loop $AB$ in the figure. The bridges are denoted by $1, 2, \cdots, n-2, n-1$. They divide the loop into $n$ equal parts. Let $l$ be

the distance between the bridges; $L_1$, $R_1$ the inductance and resistance, respectively, of each bridge; $L$, $R$, $C$, the inductance, resistance, and capacity, respectively, per mile, of the wave conductor. Let $P$ be the potential and $y$ the current at any point. $P$ is continuous and $y$ has as many discontinuities as there are bridges. In any section like $m$ we shall denote by $P_m$ and $y_m$ the potential and current at any point of the section. Denoting the distance of any point of this section from the beginning of the section by $s$, we shall have

$$P_m = V_m, \qquad y_m = x_{2m+1}, \qquad \text{when } s = l;$$

$$P_m = V_{m-1}, \qquad y_m = x_{2m}, \qquad \text{when } s = 0.$$

The problem can now be stated as follows: Find the integral of the differential equation

$$L\frac{d^2P}{dt^2} + R\frac{dP}{dt} = \frac{1}{C}\frac{\partial^2 P}{\partial s^2}$$

which will satisfy $n$ boundary conditions of the type

$$h(x_{2m-1} - x_{2m}) - 2V_m = 0,$$

where $h = ipL_1 + R_1$, and $p$ is the frequency speed of the impre-sed simple harmonic electromotive force. The solution is

$$V_m = \frac{E}{2}\frac{\sin 2(m-n)\theta}{\sin 2n\theta},$$

where

$$\cos 2\theta = \cos \mu l - \frac{\mu}{ipCh}\sin \mu l, \qquad \mu^2 = p(pL - iR),$$

$$P_{n+1} = \frac{V_n \sin \mu(l-s) + V_{n+1} \sin \mu s}{\sin \mu l},$$

$$y_{n+1} = \frac{2ipC}{\mu \sin \mu l}\{V_{n+1} \cos \mu s - V_n \cos \mu(l-s)\}.$$

When the bridges are sufficiently near with respect to the wave length which corresponds to the frequency speed of the force on the wave conductor without the bridges, then the system acts like a uniform wave conductor the capacity of which depends on the frequency. There is a critical frequency at which the wave length becomes infinite.

Professor Morley's paper is in abstract as follows: In the *Philosophical Magazine* for 1851 Sylvester gave a canonical form for a binary quantic of even order, as far as the eighth order, and suggested a general form which involved the finding of a certain covariant $V$. It seems to have been supposed that the difficulty of continuing lay in finding $V$ (see Elliott, Algebra of quantics, p. 299). An examination shows that there is no theoretic difficulty, but that the source of $V$ is easily obtainable as a determinant. The real objection to the theory lies in the complication of the expression of $V$ by means of the fundamental covariants. Thus for the quantic of the tenth order, in terms of the covariants of Sylvester's auxiliary quintic, we have, in Elliott's notation for the quintic,

$$V = 100C_{64}C_{51} - 25C_{60}C_{35} + 3C_{60}{}^2C_{55} + 120C_{71}C_{22}{}^2 + 92C_{60}C_{75} + 24C_{40}C_{22}C_{53}.$$

This formula was calculated by Mr. Coble.

Dedekind (in Supplement XI to Dirichlet's Zahlentheorie, 4th edition, p. 570) shows, if $\omega$ is any algebraic integer in a fixed realm of rationality $\Omega$, that

$$\omega^{p^f} \equiv \omega \pmod{\mathfrak{p}},$$

where $\mathfrak{p}$ is a rational prime integer that is divisible (in $\Omega$) by the prime ideal $\mathfrak{p}$ and $f$ is the degree of $\mathfrak{p}$. He then shows, pp. 570–572, that

$$t^{p^f} - t \equiv \Pi(t-\omega) \equiv \Pi P(t) \pmod{\mathfrak{p}},$$

where $t$ is a variable and where the first product is taken over a system of incongruent (mod. $\mathfrak{p}$) algebraic integers in $\Omega$ and the second product is taken over all primary

prime functions $P(x)$, whose degree is $f$ or a divisor of $f$. The investigations are restricted to algebraic numbers in a fixed realm. Professor Hancock considers instead of the algebraic numbers integral functions in any number of variables whose coefficients are the Dedekind algebraic numbers, instead of the prime ideal $\mathfrak{p}$ there occur in his general treatment prime modular systems, in whose elements enter all the variables save one. Primary prime functions are taken in which the variable that is wanting in the modular system occurs to degree $h$ and where the coefficients are integral functions of the other variables that are found in the modular system. The number of such primary prime functions whose degree is $h$ or a divisor of $h$ is calculated, and then the generalized Dedekind's theorem is developed.

Miss Wood's paper deals with the collineations of space which transform a non-degenerate quadric into itself. Starting with the theorem that such a collineation which leaves the two systems of generators invariant may be compounded of two skew reflections, the following theorems in non-euclidean geometry are deduced. For elliptic space: 1° If a straight line be displaced, the loci of the middle points of chords joining congruent points are straight lines. 2° Planes perpendicular to these chords at their middle points intersect each set in a straight line. 3° If a plane figure be displaced, the loci of the middle points of chords joining congruent points are planes. 4° Planes perpendicular to these chords at their middle points pass each set through a point. For hyperbolic space: 1° If a straight line be displaced, the locus of the middle point of chords forming congruent points is a straight line. 2° Planes perpendicular to these chords at their middle points intersect in a line. 3° If a plane figure be displaced, the middle points of chords joining congruent points lie in a plane. 4° Planes perpendicular to these chords at their middle points pass through a point in the plane. When the quadric degenerates four well known theorems of Chasles for euclidean space are obtained.

In a paper soon to be published in the *American Journal of Mathematics*, Professor Slaught has discussed the nature of invariants of a quadratic group and has shown that, inasmuch as the degree of any rational integral form is doubled under a quadratic transformation, only a rational fraction can transform into itself after throwing off a common factor in the variables from numerator and denominator. For the

quadratic Cremona group of 5! cross ratio transformations it is shown that the most general forms suitable for numerator and denominator of such invariant fractions are of degree $6n$ and throw off a factor $(z_1z_2z_3)^{2n}$ under the quadratic generator

$$z_1' : z_2' : z_3' = z_2z_3 : z_1z_3 : z_1z_2.$$

Such a form, $R_{6n}$ is found to be decomposable into two factors

$$R_{6n} = P^{2\mu} R_{6(n-2\mu)}$$

in which

$$P = z_1z_2z_3(z_1 - z_3)(z_1 - z_2)(z_2 - z_3),$$

$\mu = 0$ or a positive integer, and $R_{6(n-2\mu)}$ contains *no factor* of $P$.

The factors of $P$, set equal to zero, are the six sides of the complete quadrangle whose vertices are the four critical points of the group, and every invariant curve

$$R_{6n} = 0$$

has at each of these critical points a multiple point of order $2(n + \mu)$.

The quadratic group $G_{120}$ has a linear subgroup $G_{24}^{(1)}$ which is projectively related to the Klein's linear group of order 4!, and through a study of the invariants of these two linear groups there are found the most general invariant forms under $G_{120}$ of degree 6, 12, and 18 [$n = 1, 2, 3$], namely,

$$mA, \quad m_1A^2 + m_2P^2, \quad m_1A^3 + m_2AP^2 + m_3C,$$

in which the $m$'s are arbitrary and $A$, $P^2$, $C$ are three invariants of the binary quintic form, written in terms of the cross ratios of the five roots of the quintic. Then by a series of theorems relating the general forms $R_{6n}$ to the geometry of the group [The geometric representation was given in the October number of the *American Journal of Mathematics.*] it is shown, through a process of successive reduction, that any invariant form under $G_{120}$ is expressible as a rational integral function of $A$, $P^2$, $C$, and thus a system of fundamental forms is established in terms of which all invariant fractions under the quadratic group can be expressed.

As is known, any three solutions of the differential equation

$$\frac{\partial^2 \theta}{\partial u \partial v} + f(u, v)\frac{\partial \theta}{\partial u} + \varphi(u, v)\frac{\partial \theta}{\partial v} = 0,$$

when set equal to $x$, $y$, $z$, respectively, define a surface referred to a system of conjugate lines. Dr. Maclay determines a transformation of a character to change conjugate systems of curves on a surface into conjugate systems on the transformed surface. After some discussion of the relations of a surface and its transformed in the general case, the surfaces defined by

$$f = \varphi = \frac{n}{u + v}$$

and the corresponding transformed surfaces are more particularly studied. By suitable transformations the latter equation is reduced to the form given in the title of the paper, and the complete integral is obtained. The transformation, when employed upon the general surface of translation, will change the curves of translation into lines of curvature on the transformed surface only in the case of the right circular cylinder, and the resulting surface is any surface of revolution.

Mr. Whittemore's paper gives a short account of Hermite's investigations on quadric surfaces osculating a given surface. Hermite showed that, in general, at a finite number of simple points at any surface, the surface is osculated by all the members of a one parameter family of quadrics. In the present paper it is proved that a developable surface is osculated at every simple point by all the members of a one parameter family of quadrics. It is then shown that all the quadrics osculating a developable at any point are cones having a common vertex which is on the edge of regression of the developable. A complete geometrical definition of these cones is then obtained.

In a recent note Professor Lovett has shown that the various contact transformations of ordinary space which change straight lines into spheres are determined by pairs of bilinear equations between the point coördinates of the corresponding spaces, and that they fall into two categories according as all or certain determinants of a certain matrix reduce to constants. In the first of Professor Lovett's present papers the explicit forms of all the transformations of one category are given. There are thirty-one types, two of which depend on solutions of equations of the sixth degree; three on quintics; one on a pair of quartics; two on cubics; two are linear; and the remainder quadratic. To satisfy the equations of condition for the second cate-

gory is in general impossible. However, at least two types of $\infty^{15}$ transformations each are included in this category; they are generated by projective or dualistic transformations accompanied by Lie's transformation; they change asymptotic lines into lines of curvature.

Professor Lovett's second note solves four problems in the differential geometry of $n+1$ dimensioned space. Of the infinite groups of contact transformations of this space, the following are determined: 1° those leaving the ratios of the Gaussian curvatures of any number of tangent surfaces invariant; 2° those making the curvature of corresponding surfaces at corresponding points equal; 3° those changing applicable surfaces into such; 4° those leaving invariant the equation

$$\Sigma_1^n \Sigma p_{i,j}\, dx_i dx_j = 0.$$

In the third problem use is made of a remark of Staeckel *Comptes rendus*, 1895; the first and second generalize results obtained by Mehmke and Vivanti, *Zeitschrift für Mathematik und Physik*, 1891, 1892; *Rendiconti del Circolo matematico di Palermo*, 1891.

Peano, in *Mathesis*, volume 9, page 75 and page 110, seems to have been the first to point out that the identical vanishing of the Wronskian of $n$ functions of a single variable is not, in all cases, a sufficient condition for the linear dependence of these functions. (It is, of course, a necessary condition, provided the functions have finite derivatives of the first $n-1$ orders at every point of the region in question.) At the same time he indicated a case in which it *is* a sufficient condition, and suggested the importance of finding other cases of the same sort. Without at first knowing of Peano's work, Professor Bôcher was recently led to the same question and found a case not included in Peano's, in which the identical vanishing of the Wronskian is a sufficient condition. In the present paper, which will appear in the *Transactions*, these cases and others of a similar character are discussed.

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# ON SOME BIRATIONAL TRANSFORMATIONS OF THE KUMMER SURFACE INTO ITSELF.

BY DR. J. I. HUTCHINSON.

(Read before the American Mathematical Society, December 28, 1900.)

VERY few examples of the birational transformation of surfaces into themselves are as yet known in which the group is of infinite order.

The case of a continuous group with a finite number of parameters has been fully worked out,* but for discontinuous groups only two or three isolated examples† have up to the present time been met with.

It is in view of this, as well as on account of the general interest which attaches to the 16-nodal quartic, or Kummer, surfaces that I propose to show how to determine two groups of birational transformations of infinite order for which these surfaces are invariant.

In the first place I suppose the surface to be referred to a tetrahedron whose vertices are four nodes so chosen that none of the faces of the tetrahedron are singular tangent planes of the surface. Using homogeneous coördinates $w$, $x$, $y$, $z$, take for example the tetrahedron

$$(1) \qquad \begin{aligned} w &= a\vartheta_5^2 - b\vartheta_0^2, \quad & y &= a\vartheta_{12}^2 - b\vartheta_{34}^2, \\ x &= b\vartheta_5^2 - a\vartheta_0^2, \quad & z &= b\vartheta_{12}^2 - a\vartheta_{34}^2, \end{aligned}$$

where $\vartheta_\lambda = \vartheta_\lambda(u, v)$, and

$$a = c_{23}^2 = \vartheta_{23}^2(0, 0), \quad b = c_{14}^2.$$

The subscripts are here written according to the Weierstrass notation for the theta functions. The four functions $\vartheta_\lambda$ used in equations (1) form a Göpel quadruple and hence satify the well known Göpel biquadratic relation which I will indicate by

$$K(\vartheta_5, \vartheta_0, \vartheta_{12}, \vartheta_{34}) = 0.$$

The left member, regarded as a function $K(u, v)$ of $u$ and $v$, vanishes identically. Solving (1) for $\vartheta_\lambda$ and substituting in this relation, we obtain the required equation of the

* For an interesting account of this subject see Painlevé, Théorie analytique des équations différentielles, Paris, 1897.

† See Humbert, *Comptes rendus*, vol. 126, pp. 394, 508; and Painlevé, ibid., p. 512.

Kummer surface referred to the tetrahedron $w, x, y, z$. Writing for brevity

$$\alpha = c_{01}^2, \quad \beta = c_{34}^2, \quad \gamma = c_{12}^2, \quad \delta = c_{00}^2;$$
$$l = \alpha\beta - \gamma\delta, \quad L = lmn(\alpha\beta + \gamma\delta),$$
$$m = \alpha\gamma - \beta\delta, \quad M = lmn(\alpha\gamma + \beta\delta).$$
$$n = \alpha\delta - \beta\gamma, \quad N = lmn(\alpha\delta + \beta\gamma),$$
$$P = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)(\beta^2\gamma^2\delta^2 + \alpha^2\gamma^2\delta^2 + \alpha^2\beta^2\delta^2 + \alpha^2\beta^2\gamma^2) - 2\alpha\beta\gamma\delta(\alpha^4 + \beta^4 + \gamma^4 + \delta^4 + 4\alpha\beta\gamma\delta),$$

the required equation becomes

$$l^2m^2(w^2x^2 + y^2z^2) + n^2l^2(w^2y^2 + x^2z^2) + m^2n^2(w^2z^2 + x^2y^2) - 2L(wx + yz)(wy + xz) + 2M(wx + yz)(wz + xy) + 2N(wy + xz)(wz + xy) + 2Pwxyz = 0.$$

It is at once apparent that this equation is reproduced when we perform the birational transformation

$$(B) \qquad w' : x' : y' : z' = \frac{1}{w} : \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

Since there are 60 tetrahedra of the kind defined by equations (1), there are 60 transformations of the same type as $(B)$. These generate a group $G$ of infinite order. A much smaller number of operations, however, is sufficient to generate the same group, as we now proceed to show.

Consider the well known group $G_{16}$ of linear transformations, for which the Kummer surface is invariant. These operations either leave the tetrahedron of reference $T$ unchanged or permute it with three others. Denoting these by $T_1$, $T_2$, $T_3$, let $t_1$, $t_2$, $t_3$ represent linear transformations of $G_{16}$ which permute $T$ with $T_1$, $T_2$, $T_3$, respectively. Then $t_3 = t_1t_2$.

If $B_i$ denotes the transformation of the same type as $B$ associated with the tetrahedron $T_i$, then, since $t_i$ is of period 2, it is evident that

$$B_i = t_i B t_i.$$

It follows from this that the 60 birational transformations of the type $B$ can be generated by 15 of them properly chosen, together with the two linear transformations $t_1$ and $t_2$.

The group $G$ can be enlarged by combining with it the transformations of $G_{16}$ which leave the tetrahedron $T$ un-

changed (the faces, of course, being permuted). These transformations can be represented by

$$1,\ t',\ t'',\ t't'',$$

and the group $G$ is of index 4 under the enlarged group.

It now remains to be shown that $G$ is of infinite order. Consider the tetrahedron $w_1, x_1, y_1, z_1$, where

$$w_1 = \kappa w + \lambda x, \quad y_1 = \rho w + \sigma x + \tau z,$$
$$x_1 = \lambda w + \kappa x, \quad z_1 = \sigma w + \rho x + \tau y,$$

in which

$$\kappa = c_{01}^2 c_{03}^2, \quad \lambda = c_2^2 c_4^2,$$
$$\rho = -c_2^2 c_{01}^2, \quad \sigma = -c_4^2 c_{03}^2, \quad \tau = c_{03}^4 - c_2^4.$$

The transformation *

$$(B_1) \qquad w_1' : x_1' : y_1' : z_1' := \frac{1}{w_1} : \frac{1}{x_1} : \frac{1}{y_1} : \frac{1}{z_1},$$

written in terms of $w, x, y, z$, is

$$w' : x' : y' : z' = x : w : -\frac{\tau wx + y(\rho w + \sigma x)}{\sigma w + \rho x + \tau y}$$
$$: -\frac{\tau wx + z(\sigma w + \rho x)}{\rho w + \sigma x + \tau z}.$$

In a similar manner, by using the tetrahedron

$$w_2 = \mu w + \nu x, \quad y_2 = \xi w + \eta x + \zeta y,$$
$$x_2 = \nu w + \mu x, \quad z_2 = \eta w + \xi x + \zeta y,$$

where

$$\mu = c_4^2 c_{01}^2, \quad \nu = c_2^2 c_{03}^2,$$
$$\xi = -c_4^2 c_{03}^2, \quad \eta = c_2^2 c_{01}^2, \quad \zeta = c_{03}^4 + c_{01}^4,$$

we obtain the transformations

$$(B_2) \quad w' : x' : y' : z' = x : w : \frac{\alpha(w^2 + x^2) + \beta wx + y(\gamma x - \delta w)}{\delta x - \gamma w + \varepsilon y}$$
$$: \frac{\alpha(w^2 + x^2) + \beta wx + z(\gamma w - \delta x)}{\delta w - \gamma x + \varepsilon z},$$

where

$$\alpha = 2c_2^2 c_4^2 c_{01}^2 c_{03}^2, \qquad \beta = (c_4^4 - c_2^4)(c_{01}^4 - c_{03}^4),$$
$$\gamma = c_4^2 c_{03}^2 (c_2^4 + c_4^4), \quad \delta = c_2^2 c_{01}^2 (c_2^4 + c_4^4),$$
$$\varepsilon = (c_2^4 + c_4^4)^2.$$

* The transformation $B_1$ here indicated is not the same as the $B_1$ previously referred to.

Combining these two transformations we obtain

$$(B_1B_2) \qquad w' : x' : y' : z' = w : x : \frac{Ay + B}{Cy + D} : \frac{A'z + B'}{C'z + D'},$$

where

$$A = -\varepsilon\tau wx + (\delta w - \gamma x)(\rho x + \sigma w),$$

$$B = -[\tau wx(\delta x - \gamma w) + \alpha(\rho x + \sigma w)(w^2 + x^2) + \beta wx(\rho x + \sigma w)],$$

$$C = \varepsilon(\sigma x + \rho w) + \tau(\gamma x - \delta w),$$

$$D = (\sigma x + \rho w)(\delta x - \gamma w) + \tau\alpha(w^2 + x^2) + \tau\beta wx.$$

The question of the periodicity of this transformation is clearly the same as that of the linear fractional transformation

$$y' = \frac{Ay + B}{Cy + D},$$

where $A$, $B$, $C$, $D$ are independent of $y$ or $y'$. But in order that this transformation may have a finite period $n$ it is necessary and sufficient that*

$$(2) \qquad (A + D) = 4(AD - BC)\cos^2\frac{\lambda\pi}{n}$$

where $\lambda$ is some integer relatively prime to $n$. It is easily seen that the expressions given above for $A$, $B$, $C$, $D$ cannot satisfy such a relation as this. Hence the period of $B_1B_2$ is infinite.

If $(u, v)$ are the hyperelliptic coördinates of a point of the Kummer surface, the equations

$$w = \theta_1(u, v), \quad x = \theta_2(u, v), \; y = \theta_3(u, v), \quad z = \theta_4(u, v),$$

where for brevity $\theta_i(u, v)$ are written for the right members in (1), express the homogeneous coördinates of a point of the surface in terms of the parameter coördinates of the same point. After the transformation $B$, the coördinates can be represented in the same form

$$w' = \theta_1(u', v'), \; x' = \theta_2(u', v'), \; y' = \theta_3(u', v'), \; z' = \theta_4(u', v'),$$

where $(u', v')$ are the parameters of the point into which the point $(u, v)$ has been transformed.

A question arises as to what are the relations between $u'$,

* See Serret, Cours d'algèbre supérieure.

$v'$ and $u, v$. These cannot be algebraic, for if they were, say

$$A_1(u', v', u, v) = 0, \quad A_2(u', v', u, v) = 0,$$

then since to the point (1, 0, 0, 0) corresponds the curve of intersection of $w' = 0$ with the Kummer surface, when we substitute for $u, v$ in these relations the parameter values for this point the two equations ought to be equivalent to a single algebraic relation

$$A(u', v') = 0.$$

This latter should then be the equation of the curve which corresponds to the given point. But the equation of this curve is

$$\theta_1(u', v') = 0,$$

a relation which is not algebraic but transcendental. In fact, the relations between $u', v'$ and $u, v$ are none other than those which at once follow from the equations of transformation* $(B)$, viz.:

$$(3) \quad \theta_i(u', v')\theta_i(u, v) = \theta_j(u', v')\theta_j(u, v), \quad [i, j = 1, 2, 3, 4].$$

It would be interesting to know whether or not the group $G$ is discontinuous. It appears very likely that it is, although I have not been able to fully settle this question. It can be shown that, if $G$ contains any infinitesimal transformations, they belong to an invariant subgroup of $G$, and that they leave unchanged each point of the Kummer surface. For any combination of the generating operations $B_i$ can, according to (3), be represented in the form

$$\frac{\theta_i(u, v)}{\theta_j(u, v)} = \frac{\Phi_i(u', v')}{\Phi_j(u', v')},$$

where $\Phi_i(u', v')$ is a theta function of order $2n$, whose form it is not necessary to determine. If now

$$u' = u + \xi, \quad v' = v + \eta,$$

where $\xi$ and $\eta$ are infinitesimals, we have, on expanding,

$$\frac{\theta_i(u, v)}{\theta_j(u, v)} = F_{ij}(u, v) + \frac{\partial F_{ij}}{\partial u}\xi + \frac{\partial F_{ij}}{\partial v}\eta + \cdots,$$

* It is from a similar point of view that Humbert has remarked the existence of birational transformations of the Kummer surface into itself. See *Liouville's Jour.*, 1893, p. 466.

where

$$F_{ij} = \frac{\phi_i}{\phi_j}.$$

From this follows that

$$F_{ij}(u, v) = \frac{\theta_i(u, v)}{\theta_j(u, v)},$$

and

$$\frac{\partial F_{ij}}{\partial u}\xi + \frac{\partial F_{ij}}{\partial v}\eta = 0.$$

From the latter condition follows that the Jacobian of any two of the functions $\frac{\theta_i(u, v)}{\theta_j(u, v)}$ vanishes and hence that all are functions of one among them. But this is clearly impossible; hence if an infinitesimal transformation occurs in $G$ it leaves every point of the Kummer surface unchanged. The totality of operations in $G$ for which every point of the surface is invariant evidently form a self-conjugate subgroup of $G$. Moreover it is clear that this subgroup cannot be a finite continuous group since the Kummer surface does not enter into the category of surfaces which admit such groups.

Another group $G'$ of birational transformations of the Kummer surface into itself is determined from the fact that a one to one correspondence exists between this surface and the Weddle surface (locus of the vertex of a quadric cone which passes through six fixed points). Hence a birational transformation of the one corresponds to the like of the other. If we write

$$w : x : y : z = \vartheta_{01}\vartheta_{12}\vartheta_5 : \vartheta_{23}\vartheta_{03}\vartheta_5 : \vartheta_{23}\vartheta_{12}\vartheta_4 : \vartheta_{03}\vartheta_{01}\vartheta_4$$

$$a : b : c : d = c_{01}c_{12}c_5 : c_{23}c_{03}c_5 : c_{23}c_{12}c_4 : c_{03}c_{01}c_4,$$

$$\alpha : \beta : \gamma : \delta = c_{23}c_{03}c_4 : c_{01}c_{12}c_4 : c_{01}c_{03}c_5 : c_{12}c_{23}c_5,$$

the equation of the Weddle surface is

$$\begin{vmatrix} xyz & w & a & \alpha \\ wyz & x & b & \beta \\ wxz & y & c & \gamma \\ wxy & z & d & \delta \end{vmatrix} = 0.$$

This equation is unchanged for a transformation of the same form as $(B)$. I will denote this transformation by $C$. Since the equation of the surface can be written in 15 dif-

ferent ways in this form, we have 15 corresponding transformations which generate a group of infinite order. For consider the tetrahedron $w_1, x_1, y_1, z_1$, where

$$w_1 = w, \quad (b+a)x_1 = bw - ax,$$
$$(c+a)y_1 = cw - ay, \quad (d+a)z_1 = dw - az.$$

The associated transformation is

$$w_1' : x_1' : y_1' : z_1' = \frac{1}{w_1} : \frac{1}{x_1} : \frac{1}{y_1} : \frac{1}{z_1},$$

or

$$w' : x' : y' : z' = \frac{1}{w} : \frac{bx + (a+2b)w}{awx - bw^2} : \frac{cy + (a+2c)w}{awy - cw^2}$$
$$: \frac{dz + (a+2d)w}{awz - dw^2}.$$

Denoting this transformation by $C_1$ we have for $C_1C$

$$w' : x' : y' : z' = w : \frac{(a+2b)wx + bw^2}{aw - bx} : \cdots$$

Here $x'$ is proportional to an expression of the form

$$\frac{Ax + B}{Cx + D}.$$

It is easily seen that

$$\frac{(A+D)^2}{AD - BC} = 4,$$

and hence the condition (2) reduces to

$$\cos^2 \frac{\lambda\pi}{n} = 1.$$

But since $0 < \lambda < n$ this relation cannot be satisfied for a finite value of $n$. Hence the transformation $C_1C$ has an infinite period.

CORNELL UNIVERSITY,
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# THEOREMS CONCERNING POSITIVE DEFINITIONS OF FINITE ASSEMBLAGE AND INFINITE ASSEMBLAGE.

BY MR. C. J. KEYSER.

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THE well known wide disagreement* among workers in assemblage theory as to the meaning that should be attached to certain indispensable terms, such as assemblage, defined assemblage, given assemblage, law of formation, and so on, indicates alike the need and the possibility of a more critical grounding of this doctrine. Here as elsewhere the guiding principle of criticism should be the principle of Parsimony, the so-called Razor of Occam: Entia non sunt multiplicanda præter necessitatem. For the problem is one of simplification, of logical economy, of minimizing hypothesis. The ideal is, in a word, to dispense with the undefined. Now nothing seems clearer than that no amount of criticism, however acute, can completely eliminate the undefined from the foundations of knowledge. For every explicit involves some implicit, and so there must be not merely assumption, but implicit and therefore undefined assumption. Whence appears that a residuum of indetermination and doubt must elude even the most refined and searching analysis, and ultimate simplicity, perfect certainty, absolute rigor, remain beautiful dreams, destined never to be quite realized. Nevertheless the problem admits of closer and closer approximate solution, which, theoretically at least, takes the form of an unending series, namely, of substitutions of simpler (there is no simplest) undefined for the less simple. Unfortunately the process appears to conduct sooner or later from seeming light and certainty into the "frightful shadow-land of metaphysics." But it is only appearance and only seeming, for it is the common lot, consciously or unconsciously, to dwell in that land always. Shadows critical reflection undoubtedly produces, but shadows are incident to the illumination of darkness. Cousin is right: La critique est la vie de la science.

*Cf. J. Tannery: "De l'infini mathématique," *Revue générale des sciences*, vol. 8 (1897). Dedekind: Was Sind und Was Sollen die Zahlen. § 3, Art. 32; § 5, Art. 64. G. Cantor: "Sur les ensembles infinis et linéaires de points," *Acta Math.*, vol. 2. Couturat: De l'infini mathématique, Appendice, Note IV. E. Borel: Leçons sur la théorie des fonctions, Chap. I. J. Tannery: Introduction à la théorie des fonctions d'une variable, § 15.

This paper presents some results of an attempt, consistently with the mentioned principle of criticism, to show that definitions, of a more general character than those usually considered to be mathematically available, are in fact sufficient for the establishment of certain fundamental propositions of assemblage theory.

## I. *Hypotheses, and Definition of Terms.*

It is assumed that, if there be anything which throughout some definite time is intrinsically regardable as an individual, as one, this intrinsic capability persists throughout every definite time. Any such thing will be named *element*. It is further assumed: (1) that there are elements; (2) that one of the elements is the idea $\epsilon$, that each of the elements, including $\epsilon$, *is* an element; (3) that, if two elements be distinguishable, the mark of distinction is an element.

The element $\epsilon$, serving as a bond of union of all elements, strictly defining all as an individual, as the sole invariant content of the element category, imparts sufficient precision to the expression totality of elements. This totality, which might be called the universe of elements, may be denoted by $U$. $U$ being, by definition, itself an element, is to be regarded as contained in itself.

Two elements are *identical* if they are indistinguishable from each other. Two such elements are, for all logical purposes, for attention, not two but one; indeed, they can not even be counted as two, for to suppose them so countable is to suppose them capable of arrangement in the time-order of before and after, which is tantamount to supposing them distinguishable. Hence $U$ contains no identical elements: *there are no identical elements.*

An element that is two or more elements regardable as an individual may be called a *composite* element; in the contrary case, an *ultimate* or *simple* element.

Now let $e$ and $e'$ be any two elements. There is at least one mark of distinction. Represent it by the prime $'$. This mark is, by (3), itself an element. Consequently $e'$ is a composite element. Similarly for $e$. Like reasoning applied to the element $'$ will show it to be composite, and so on indefinitely; indefinitely, for if the process exemplified could terminate, it would end with an element having no mark of distinction, *i. e.*, with a mark of distinction itself without distinction. Such a mark would be an element identical with some (in fact every) other. Hence all elements are composite: *there are no simple or ultimate elements.*

It thus appears that an element is a shield with two sides, one fact $F$ with two phases: the phase of being constituted of elements, constituents; and the phase of being a bond uniting the latter into one, an individual. The two phases of $F$ may be briefly called respectively the *multi*-phase and the *uni*-phase. Neither is logically prior. Either may be regarded as *primary*, the other as *secondary*—the choice to be determined by considerations of expedience. $F$ will be called element or assemblage (of elements) according as the uni-phase or the multi-phase of $F$ be taken as primary. The notions, element and assemblage, are, therefore, *reciprocal** and complemental, each is implicit in, incomplete without, determined by, the other.

This principle of reciprocity enables us to say: all elements and assemblages are *defined*. If $F$ be an assemblage, its definition is its uni-phase, its element-character; if $F$ be thought as an element, its definition is its multi-phase, its character as a certain composite. For easier comparison of this with the corresponding conceptions of Cantor and Dedekind,† we state it at length thus: Every assemblage is defined in the sense that the principle of excluded middle validates, with respect to any assemblage, the proposition: any element whatever is a constituent of the assemblage or it is not. Reciprocally, every element is defined in the sense that the same principle justifies, in regard to any element, the proposition: the constituents of any assemblage whatever compose the element or they do not.

The term "defined" is here used in its *objective*, which is at the same time its fundamental and etymological, sense of bounded or marked off from. Every such objective definition is an $F$ with its two phases. Its essence is being, not actually regard*ed* but merely regard*able*. When actually regarded, made the recipient of attention, then, lying in consciousness or expressed in language or both, it becomes an ordinary, *subjective*, definition. When so expressed the corresponding $F$ (assemblage or element) may be said to be *given*. Discussion of the exceeding difficult question‡ of "given" is reserved for a future communication as is also a fuller exhibition of the cardinal importance of the principle of recipro-

* It is noteworthy that the so-called "principle of reciprocity" which manifests itself familiarly in various specialized and concrete ways, notably in geometry, is in all strictness a *principle* and is such for assemblage theory in general, being inherent, as just shown, in the inmost core of the discipline.

† Cf. Cantor and Dedekind: *Op. cit.*

‡ In relation to this matter, the first of J. Tannery's essays already cited may be consulted with much profit.

city. In regard to the latter we merely add here the now rather obvious facts: that assumptions (1), (2), and (3) involve like assumptions respecting assemblages; that there are no identical assemblages; that, as every assemblage has its uni- and every element its multi-phase, there is one and but one element for each assemblage and reciprocally; and, that, in particular, $U$ plays the double rôle of being at once the totality of elements and the totality of assemblages, including itself in both characters.

If $E$ and $E'$ are any two assemblages such that there is an assemblage $M$ having the properties: (1) each element of $M$ is a pair of elements $e$ and $e'$, where $e$ and $e'$ signify element of $E$ and element of $E'$ respectively; (2) each element $e$ occurs once and but once, and no element $e'$ occurs twice, in $M$; the assemblage $E$ may be said to be *projectible* upon the assemblage $E'$. As in $M$ each element $e$ of $E$ is associated with a certain element $e'$ of $E'$, the assemblage $M$ may be said to indicate a mode or *manner* of the projectibility; *a* rather than *the* manner, because obviously, if $E$ be projectible upon $E'$ in one manner, it will in general be projectible upon $E'$ in more than one manner. If, in a given $M$, $e_k'$ of $E'$ is associated with $e_k$ of $E$, $e_k'$ may be called the *projection* of $e_k$ for the given $M$. The totality of such projections is an assemblage which may be denoted by $P$.

If two assemblages $E$ and $E'$ are each projectible upon the other in a manner $M$, $E$ and $E'$ may be said to be *congruent* in the manner $M$, or a unique and reciprocal correspondence may be said to exist in the manner $M$ between $E$ and $E'$. If two assemblages $E$ and $E'$ be congruent, the relationship may be stated symbolically by writing

$$E \equiv E'.$$

If every element of an assemblage $A$ is an element of an assemblage $A'$ and if one or more elements of $A'$ are not in $A$, $A$ will be called a *part* (partie intégrante of Cantor, echter Theil of Dedekind) of $A'$, and we may write

$$A < A'.$$

If this relation subsists between $A$ and $A'$, then plainly $A$ is projectible upon $A'$. Also any $E$ is congruent with itself. The equation

$$A_1 + A_2 = A$$

may be employed to indicate at once that $A_1$ and $A_2$ are each a part of $A$, that $A_1$ and $A_2$ have no element in common, and that every element of $A$ is an element of $A_1$ or of $A_2$.

II. *Demonstration of Theorems.*

Let $E$ be an assemblage. Either $E$ is congruent with an assemblage

$$(N) \qquad 1, 2, \cdots, n,$$

where $n$ is some definite integer, or such is not the case. If $E$ be not congruent with $(N)$, the assemblage

$$(N') \qquad 1, 2, \cdots, n, \cdots$$

is projectible upon $E$. To prove this it is sufficient to prove that, however large a value $n'$ of $n$ be chosen in advance, the assemblage

$$1, 2, \cdots, n', n' + 1$$

is projectible upon $E$. That this assemblage is projectible upon $E$ follows immediately from the fact that, were it not so projectible, $E$ would itself be projectible upon $(N)$ for $n \overline{\geqq} n'$, which is contrary to hypothesis. Let the assemblage

$$(M) \qquad (1, e_1), (2, e_2), \cdots, (n, e_n), \cdots$$

indicate a manner of projectibility of $(N')$ upon $E$. Then the assemblage $P$ of projections will be

$$e_1, e_2, \cdots, e_n, \cdots$$

and we may write

$$E = P + R$$

where $R$ denotes the assemblage of those elements of $E$ which, if there be any such, are not elements of $P$. Represent the assemblage

$$e_2, e_4, \cdots, e_{2n}, \cdots$$

by $P_1$. It is plain that $P$ is projectible upon $P_1$ in the manner

$$(M_1) \qquad (e_1, e_2), (e_2, e_4), \cdots, (e_n, e_{2n}), \cdots$$

and $R$ is projectible upon itself; hence the assemblage $P + R$ is projectible upon the assemblage $P_1 + R$; but

$$P + R = E \quad \text{and} \quad P_1 < P$$

whence

$$P_1 + R < E.$$

$E$ is therefore projectible upon a part of itself. As any assemblage is obviously congruent with the assemblage of its projections, $E$ is seen to be congruent with a part of itself. Accordingly we may state the

THEOREM ($a$). *If any assemblage $E$ be such that there is no definite value of $n$ for which the assemblage*

$$1, 2, \cdots, n$$

*is congruent with $E$, then $E$ is congruent with a part of itself.*

From this follows directly the contrapositive

THEOREM ($b$). *If any assemblage $E$ be not congruent with a part of itself, then, for some definite value of $n$, the assemblage*

$$1, 2, \cdots, n$$

*is congruent with $E$.*

Are the converses of these propositions correct? Suppose that

$$A < E, \text{ and that } E \equiv A \text{ in some manner } M.$$

It is required to determine whether there is a determinate integer $n$ such that the assemblage

$$(N) \qquad 1, 2, \cdots, n$$

is congruent with $E$.

Let $e_k$ denote any definite element of $E$. In $M$ this element is paired with some definite element $e_j$ of $E$, whence $e_j$ is an element of $A$. Now, if $E$ is congruent with $(N)$ in one or more manners, let $M'$ stand for any definite one of them. In $M'$ the element $e_k$ has for its projection, or is paired with, some definite integer $n'$ of $(N)$ where of course it is undetermined whether $n'$ is less than or equal to $n$. Accordingly, $M$ and $M'$ together furnish a scheme for associating in a one to one fashion the elements of $A$ with those of $(N)$: given any element, as $e_k$ of $E$, those two elements, $e_j$ and $n'$, of $A$ and $(N)$, respectively, shall be associated, which are associated with $e_k$ in $M$ and $M'$, respectively. Denote by $M''$ the assemblage of such pairs $e_j$, $n'$. It thus appears that if

$$E \equiv A$$

in some definite manner $M'$, then

$$A \equiv (N)$$

in a definite related manner $M''$. But since

$$A < E,$$

there is at least one element $e_i$ of $E$ which is not an element of $A$, and, therefore, no one of such elements $e_i$ enters a pair in $M''$. Hence $A$ is not congruent with $(N)$ in the

manner $M''$; whence follows that $E$ is not congruent with $A$ in the manner $M'$, and, therefore, not in any manner.

We have, accordingly, the following propositions:

THEOREM ($a'$). *If any assemblage $E$ is congruent with a part of itself, there is no assemblage*

$$1, 2, \cdots, n \qquad (n \text{ definite})$$

*with which $E$ is congruent.*

THEOREM ($b'$). *If there is an assemblage*

$$1, 2, \cdots, n \qquad (n \text{ definite})$$

*with which an assemblage $E$ is congruent, $E$ is not congruent with a part of itself.*

These reciprocally contrapositive propositions are the converses, respectively, of ($a$) and ($b$).

## III. *Logical Significance of the Four Theorems.*

Denote by $U$ the assemblage of all assemblages, including $U$. The elements of $U$ are the component assemblages of $U$. Of these there are two, $F_1$ and $I_1$, such that the former signifies the assemblage of all assemblages which are not, and the latter signifies the assemblage of all assemblages which are congruent each with a part of itself. By the principle of excluded middle,

$$U = F_1 + I_1.$$

By the same principle,

$$U = F' + I',$$

where $F'$ denotes the assemblage of all assemblages of which each has, and $I'$ denotes the assemblage of all assemblages of which none has, the property of being projectible upon (or congruent with) an assemblage

$$1, 2, \cdots, n \qquad (n \text{ definite}).$$

Now the logical import of the propositions ($a$), ($b$), ($a'$), and ($b'$) is that $F_1$ and $F'$, as also $I_1$ and $I'$, are identical, element for element. Accordingly, we may define as follows:

(1) *A finite assemblage is one which is not congruent with any part of itself.*

(2) *An infinite assemblage is one which is not congruent with any assemblage.*

$$1, 2, \cdots, n \qquad (n \text{ definite}).$$

Or quite equivalently thus:

(3) *A finite assemblage is one which is congruent with some assemblage*

$$1, 2, \cdots, n \qquad (n \text{ definite}).$$

(4) *An infinite assemblage is one which is congruent with a part of itself.*

In neither pair of definitions are the notions finite and infinite *formally* represented as mutually exclusive or as negatives of each other; while the second pair has the additional advantage of representing both the notions in question as *positive** concepts.

Before passing to the final section of this paper, it seems worth while to pause long enough to glance, from the present point of view, at the notion, cardinal number of a finite assemblage, and to offer a proof that this notion, for any given finite assemblage, possesses the property of invariance.

Consider the *ordered* assemblage

$$(S) \qquad 1, 2, \cdots, n', \cdots, n, \cdots n'',$$

where $n'$, $n$ and $n''$ are definite and subject to the condition that, in $(S)$, $n'$ comes before $n$ and the latter before $n''$. Denote by $S_{n'}$, $S_n$, and $S_{n''}$ the assemblages obtained by supposing in succession that $(S)$ ends with $n'$, $n$ and $n''$, and in each case abstracting from the order of the elements. $S_{n'}$, $S_n$, and $S_{n''}$, being congruent each with itself, are finite assemblages, and, as by definition

$$S_{n'} < S_n \quad \text{and} \quad S_n < S_{n''},$$

we have by (3)

$$\text{neither } S_{n'} \equiv S_n \text{ nor } S_n \equiv S_{n''}.$$

Now let $E$ be any finite assemblage. By (3) there is some $S_n$ for which

$$E \equiv S_n.$$

We now *define* $n$ to be the cardinal number of the assemblage $E$. Plainly, this number is independent of every arrangement of the elements of $E$; for if, by any particular arrangement,

$$E \equiv S_{n'} \text{ or } E \equiv S_{n''},$$

we would have also

* As to this phase of the matter and also in relation to definitions (1) and (4), cf. Jules Tannery, *op. cit.*; Dedekind, *op. cit.*; G. Cantor, *op. cit.*; Couturat, *op. cit.*

$$S_n \equiv S_{n'} \text{ or } S_{n''} \equiv S_n.$$

It readily follows that any two congruent finite assemblages have the same cardinal number.

IV. *Connection with the Logic of Relatives: Another Point of View.*

Consider the argument:

Every element of an assemblage $E$ is projected upon some other element of $E$;

No element of $E$ is the projection of more than one element of $E$;

$\therefore$ Every element of $E$ is a projection of an element of $E$; *i. e.*, the assemblage of projections is congruent with $E$.

This argument is of the type of the so-called syllogism of transposed quantity, a mode of ratiocination of which the earliest scientific account is found in the Logic of De Morgan. Its uses and limitations have been pointed out by Mr. C. S. Peirce,* to whom is due also its perfection as an instrument of research as well as the indication of its significance in modern logic as such. From definition (1) or from the last proposition of section III, it is seen that the conclusion is warranted by the premises always when and only when $E$ is a finite assemblage. Reversing the order of ideas, the above argument, regarded as purely a *form*, may be taken as the 'logical' discriminant of the assemblage

$$U = F_1 + I_1$$

and accordingly we might define: any given assemblage is finite or infinite, belongs to $F_1$ or to $I_1$, according as the syllogism is valid or not valid for that assemblage. Such appears to be in fact Mr. Peirce's point of view in one or two instances. At all events, it requires merely verbal substitution to render the syllogism in question quite identical with the "Hottentot" and "Texan" arguments employed by him in the works cited below. The interest here attaches, of course, to the demonstrated ultimate equivalence of the two points of view.

COLUMBIA UNIVERSITY.

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* Peirce: "On the Algebra of Logic," *Amer. Jour. of Math.*, vol. 7, p. 201. "On the Logic of Number," *Amer. Jour. of Math.*, vol. 4, p. 75. Definition of "finite," Century Dictionary.

# DINI'S METHOD OF SHOWING THE CONVERGENCE OF FOURIER'S SERIES AND OF OTHER ALLIED DEVELOPMENTS.

BY MR. WALTER B. FORD.

(Read before the American Mathematical Society, December 28, 1900.)

IF it is required to show that the arbitrary function $f(x)$ of the real variable $x$ may be developed into a Fourier series for all values of $x$ lying between $-\pi$ and $\pi$ ($-\pi$ and $\pi$ at most excluded) then, remembering that for any particular value of $x$, such as $x=a$, the sum of the first $n+1$ terms of the series is

$$(1) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\frac{\sin\frac{2n+1}{2}(x-a)}{\sin\frac{1}{2}(x-a)}dx,$$

it is easily shown that we need merely examine the limits approached by the integrals

$$(2) \qquad \int_{e}^{d} f(x)\frac{\sin\frac{2n+1}{2}(x-a)}{\sin\frac{1}{2}(x-a)}dx,$$

$$(3) \qquad \int_{e}^{d} f(x)\frac{\sin\frac{2n+1}{2}(x-a)}{\sin\frac{1}{2}(x-a)}dx,$$

as $n$ increases indefinitely, $e$ and $d$ being any numbers such that $0<e<d\leqq\frac{1}{2}\pi$. In case these limits exist and have certain simple properties it becomes evident that the given series will be convergent for any value $x=a$ which lies between $-\pi$ and $\pi$, and will have as its sum either $f(a)$ or

$$\frac{f(a+0)+f(a-0)}{2},$$

while at either of the points $x=-\pi$ or $x=\pi$ the sum will be

$$\frac{f(-\pi+0)+f(\pi-0)}{2}$$

Moreover, a closer examination of integrals (2) and (3) shows that if $f(x)$ satisfies any one of certain sets of preliminary conditions (of which Dirichlet's conditions are a special case), then these integrals (2) and (3) possess the required properties; for in such cases the problem reduces to the establishing of certain well known properties of the more simple integrals

$$\int_0^d \frac{\sin \frac{2n+1}{2}(x-a)}{\sin \frac{1}{2}(x-a)}\, dx, \tag{4}$$

$$\int_c^d \frac{\sin \frac{2n+1}{2}(x-a)}{\sin \frac{1}{2}(x-a)}\, dx, \tag{5}$$

where $c$ and $d$ have the same significance as before. In such cases, then, the original problem reduces to questions which are entirely independent of the function $f(x)$ which was to be developed. Now, integrals (4) and (5) being once closely examined and their properties which are essential to the present problem being once tabulated, the question arises whether there are not still other functions of $x$, $n$ and $a$, which like

$$\frac{\sin \frac{2n+1}{2}(x-a)}{\sin \frac{1}{2}(x-a)},$$

when integrated between certain determined or undetermined limits $(0, d')$ in the one case and $(c', d')$ in the other, will possess the same essential properties which belong to (4) and (5), and, as thus integrated, will therefore form integrals bearing the same relation to *new* series developments of $f(x)$ for $x$ between certain limits $(a, b)$ that (4) and (5) bore to the Fourier development of $f(x)$ for $x$ between $-\pi$ and $\pi$. Evidently if such functions exist, representing any one of them by $\varphi(x-a, a, n)$, or better by $\varphi(x-a, a, h_n)$ where $h_n$ is any particular expression which (like the $\frac{2n+1}{2}$ which appears in (4) and (5)) is always positive and increases indefinitely with $n$, the integrals in question will be

$$\int_0^{d'} \varphi(x-a, a, h_n)dx, \tag{6}$$

$$\int_{c'}^{d'} \varphi(x-a, a, h_n)dx, \tag{7}$$

and for any particular value of $x$ such as $x = a$ it is evident that the sum of the first $n + 1$ terms of this new series will be an integral corresponding to the integral (1), and in fact it may be easily shown that this sum will be

$$\frac{1}{2G}\int_a^b f(x)\varphi(x - a, a, h_n)dx, \tag{8}$$

where $G$ is a determinate constant independent of $x$ and of $a$ and different from zero.

Dini begins by showing that there are an infinite number of such functions $\varphi(x - a, a, h_n)$*. Then considering the series

$$\frac{1}{2G}\int_a^b f(x)\varphi(x - a, a, h_0)dx \tag{9}$$

$$+ \frac{1}{2G}\sum_1^\infty \int_a^b f(x)\{\varphi(x - a, a, h_n) - \varphi(x - a, a, h_{n-1})\}dx,$$

the sum of whose first $n + 1$ terms is evidently the above expression (8), he proceeds to make farther assumptions as regards the functions $\varphi(x - a, a, h_0)$, $\varphi(x - a, a, h_1)$, $\cdots$ $\varphi(x - a, a, h_n)$; *i. e.*, he assumes that each of these functions is such that every term of (9) may be expressed in a certain typical form (which is the form common to all known developments). One of the chief requirements of a term in this form is that it shall involve in a specified way certain special functions of $x$, or of $x$ and $\lambda_n$, where $\lambda_n$ is a parameter variable only with $n$, which functions we may suppose to have taken in advance as the functions in terms of which any new development might be desired (these functions in the case of the Fourier series being $\sin nx$ and $\cos nx$.)

This being done, it is evident that if $f(x)$ satisfies any one of the sets of conditions mentioned before, we may now form a sufficient condition that this function may be developed for all values of $x$ lying between the determined or undetermined limits $(a, b)$ ($a$ and $b$ at most excluded) in terms of given functions of $x$, or of $x$ and $\lambda_n$, the development being of the form common to all known developments (developments such as occur, for example, in the general study of mathematical physics).

This condition is described by saying that the sum of the first $n + 1$ terms of the given series should be equal to an

* Reference is here made to Dini's work entitled: Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale; Pisa, 1880.

integral of the form (8), where $\varphi(x-a, a, h_n)$ is a function of the type mentioned above. But to use this condition as a criterion in any case where we desire to test a given development is evidently difficult, for we are thus required to determine certain facts about the nature of an infinite series whose individual terms are generally of a complicated character. As an aid to this undertaking, Dini now employs an ingenious method due in substance to Cauchy. He virtually shows how a function of the complex variable $z$ may be constructed so that it will have an infinite number of poles of the first order distributed at finite distances from each other along the positive side of the axis of reals, this function being otherwise monogenic within finite regions of the $z$ plane, and being, moreover, so constructed that its residua in these poles (or at least in some of them) are equal respectively to the individual terms of the given series in $x$ which we are to test. This once accomplished, it is evident that we have merely to integrate this function of $z$ about any closed contour lying to the right of the axis of pure imaginaries and enclosing the first $n+1$ of these poles and not passing through any one of them, in order to obtain readily in the form of a definite integral (in the complex) the sum of the first $n+1$ terms of the given series; and by enlarging the contour so as to enclose more and more of the poles we shall have in the limit as the contour thus enlarges indefinitely an expression in the form of a definite integral (in the complex) which actually represents the sum of the given series. Thus, by this means our investigation is transferred from questions regarding the given series to questions regarding a definite integral which varies with $n$, and for any special value of $n$ is equal to the sum of the first $n+1$ terms of the given series. Whenever this integral reduces to one of the integrals (8) we are assured that the given development is possible. To study this reduction is, however, difficult in many cases, but Dini shows how conclusive results may be obtained in the case of Fourier series, series in terms of zonal harmonics, Bessel functions, and elliptic functions, and he makes it evident that the method leads to decisive results in many other cases.

ANN ARBOR,
*December*, 1900.

## SHORTER NOTICES.

*Application de la Méthode Vectorielle de Grassmann à la Géométrie Infinitésimale.* Par HENRI FEHR.* Georges Carré et C. Naud, Éditeurs. Paris, 1899. 94 pp.

THE mathematicians Grassmann and Hamilton who almost simultaneously published their first work upon a "calculus of space" were undoubtedly geometers, not analysts—abstract and speculative they may have been—but still geometers. We have only to regret that they were not also clear stylists; for it must be admitted that, with the exception of the attempts of certain persons to show how everything may be done by the methods of vector analysis or quaternions, nothing could be more fatal to the popular acceptation and use of a space analysis than the form in which it was presented to the public by the inventors. Succeeding writers for the most part seem to have erred along the same lines or to have forgotten the stress originally laid upon the interpretation of the analysis. It is therefore with great pleasure that we read M. Fehr's little book, which is written with such admirable clearness and selected with such tasteful care that in the compass of ninety-one pages there is included, without the slightest suspicion of crowding, a preface, an introduction on the use of vector analysis, and a fairly complete treatment of differential geometry.

M. Fehr originally wrote his book as a thesis to be presented for the degree of doctor of science at the University of Geneva. As a thesis, the work contains nothing original either in vector analysis or in geometry. All the results and methods were known well enough before. Indeed, anyone who has heard such lectures as are given at our leading universities upon these two branches of mathematics ought to be able to put together the material in this book with almost no difficulty. But to put it together in so pleasing a manner is a far harder task. The reviewer, therefore, must deal with the manner of presentation rather than with that which is presented.

The votaries of vector analysis will find in M. Fehr's book little to cheer them. In fact, so meager is the development of the subject that one might almost say the entire analysis consists of three symbols

* Now professor of geometry and algebra at the University of Geneva.

$[a_1\ a_2] = Va_1\ a_2$, the vector product of two vectors,
$[a_1 \mid a_2] = -Sa_1\ a_2$, the scalar product of two vectors,
$[a_1\ a_2\ a_3] = -Sa_1\ a_2\ a_3$, the scalar product of three vectors.

These expressions are used, to all intents and purposes, as abridged notations. If two vectors $a_1$ and $a_2$ are perpendicular, the fact is indicated by writing $[a_1 \mid a_2] = 0$; if they are parallel, by $[a_1\ a_2] = 0$; and if three vectors $a_1$, $a_2$, $a_3$ are coplanar, by $[a_1\ a_2,\ a_3] = 0$. The working rules of the analysis and the formal transformations of the symbols are developed scarcely at all. M. Fehr knows very well what to omit, and he omits a great deal. The differentiating operator $\nabla$, which can be used with considerable advantage in some portions of differential geometry, is not introduced at all. This omission tends to clearness and simplicity. The linear vector function is also left out to advantage. Indeed, it seems as if the formal methods of vector analysis are none too well suited to the development of differential geometry. The vectors appear more or less artificial. They are not thrust upon us irresistibly, as in physics. The need of a vector-analysis is not so apparent as it might be. M. Fehr, therefore, has done well to introduce as little as possible of the vector machinery. Perhaps it is well to note that M. Appell has followed a similar course in his treatment of the theory of vectors: Traité de mécanique rationelle, volume I, chapter I.

The presentation of the elements of differential geometry given by M. Fehr is hardly to be excelled. The usual important and fundamental results are obtained in much less space and time than is customary, and yet with perhaps a gain instead of a loss in clearness. The reason for this unusual brevity is not so much the compactness of the vector notation as the concreteness of the vector idea. No time is wasted in developing analytical formulæ. The mind is brought to bear directly upon the geometric questions at hand, and they are solved. It is this constant appeal to visualization that shortens the work; it is this that distinguishes the book from others; it is this that adds so much of perspicuity; and it is this that leads us to recommend the book most heartily to all who teach or study this subject. The vector ideas will clarify and render definite the conceptions of geometry to an extent scarcely possible with other methods. The insignificant amount of vector analysis need cause no fear of difficulty.

Mr. Fehr is to be thanked for his excellent book, and Messrs. Carré et Naud for the clear and almost errorless

typography. The following reproduction of the table of contents will serve to show more satisfactorily than anything else the scope of the work: *Préface, Introduction:* Rappel de quelques notions de calcul géométrique; *Des courbes gauches:* 1° Généralités, 2° Courbure et rayon de courbure, 3° Torsion, formules de Frenet, 4° Courbure normale, formule de Lancret; *De la théorie des surfaces:* 1° Généralitiés, 2° Relations fondamentales; *De la courbure des courbes tracées sur une surface:* 1° Théorème de Meusnier, 2° Courbure des sections principales, 3° Formule d' Euler; *De la courbure des surfaces:* 1° Courbure totale, application aux surfaces réglées, 2° Courbure moyenne; cas particulier, 3° Courbure moyenne quadratique; *Des lignes tracées sur une surface;* 1° Systèmes conjugués, 2° Lignes de courbure; théorème de Dupin, 3° Lignes asymptotiques, 4° Lignes géodésiques; courbure géodésique d'une ligne tracée sur une surface.

E. B. WILSON.

*Annuaire pour l'An 1901, publié par le Bureau des Longitudes.* Paris, Gauthier-Villars.

THE new *Annuaire* which has just been received contains this time but few alterations. In the astronomical portion, however, the article on the various kinds of calendars has been rewritten and enlarged, the rules for obtaining the date of Easter in any year being much improved in statement and completeness. The historical part containing the tables of small planets, comets, etc., is, of course, brought up to date. A useful addition is a curve showing graphically the amount of the equation of time for any period of the year; this has been carefully drawn by M. Cornu. In the scientific notices, M. Cornu resumes his article of last year under the title 'Le transport électrique de la force'; M. Poincaré gives the report of the committee appointed to consider the question of the revision of the Quito meridian arc; M. Bassot writes the history of the introduction and establishment of the metric system. MM. Loewy and Bouquet de la Grye respectively contribute accounts of the international astronomical and geodetic congresses held in Paris during the summer; the aeronautical congress is represented by M. J. Janssen's presidential address. The last named also gives his annual report of the work done in the Mont Blanc Observatory during the past year. The final notice contains the oration of M. le commandant Guyou delivered at the meeting held to commemorate the late M. de Bernardières.

The increase in size of the *Annuaire* is due to the larger number of scientific notices and to the insertion of nearly 100 pages of advertisements. The annual proper is only eight pages longer than its predecessor.

ERNEST W. BROWN.

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## ERRATA.

On page 121 of the present volume of the BULLETIN, in the last line of my paper, the word *column* should be changed to *row*. The slip of my pen which produced this error is the more to be regretted as it makes Peano's theorem appear almost identical with the theorem which I had stated above, whereas the two theorems are wholly distinct from each other.

MAXIME BÔCHER.

Other errata in the present volume which have come to the attention of the editors are:—

Page 11, line 3 and line 30, *for* conformably *read* conformally.

Page 136, line 27, *for* $p$ *read* $p_{y'}$.

Page 142, line 22, *for* $r_1$ *read* $r_1^2$.

Page 158, line 15, *for* pure by *read* purely.

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## NOTES.

A NEW Annual Register of the AMERICAN MATHEMATICAL SOCIETY has just been published and distributed among the members. Copies of the Register may be obtained from the Secretary.

THE first (January) number of Volume II of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Invariants of systems of linear differential equations," by E. J. WILCZYNSKI; "Divergent and conditionally convergent series whose product is absolutely convergent," by FLORIAN CAJORI; "Sets of coincidence points on the non-singular cubics of a syzygetic sheaf," by M. B. PORTER: "Note on non-quaternion number systems," by W. M. STRONG; "On the reduction of the general abelian integral," by J. C. FIELDS; "Ueber Flächen von constanter Gauss'scher Krümmung," by DAVID HILBERT; "Note

on the functions of the form $f(x) \equiv \varphi(x) + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$ which in a given interval differ the least possible from zero," by H. F. BLICHFELDT.

THE January number of the *Annals of Mathematics* contains the following papers: "Extension of Hurwitz's proof for the transcendence of $e$ to the transcendence of $\pi$," by R. E. MORITZ; "An application of elliptic functions to Peaucellier's link-work (inversor)," by A. EMCH; "Note on the geometrical treatment of conics," by C. A. SCOTT; "On two-dimensional fluid motion through spouts composed of two plane walls," by R. A. HARRIS; "On a special class of abelian groups," by G. A. MILLER; "The theory of linear dependence," by M. BÔCHER; "Brilliant points of a family of concentric spheres," by Lieut. A. HAMILTON; "Multiply perfect numbers," by D. N. LEHMER.

AT a meeting of the London Mathematical Society held on December 13, 1900, the following papers were read: Mr. A. B. BASSET: "The real points of inflexion of a curve"; Miss M. E. BARWELL: "On the conformal representation of polygons on a half plane"; Professor E. B. ELLIOTT: "The syzygetic theory of orthogonal binariants"; Mr. A. L. DIXON: "An addition theorem for hyperelliptic functions"; Professor W. BURNSIDE: "On some properties of groups of odd order, II"; Mr. R. W. HUDSON: "On discriminants and envelopes of surfaces"; Mr. H. W. RICHMOND: "Note on the inflexions of curves with double points."

THE British association for the advancement of science will hold its next annual meeting at Glasgow, beginning on September 11. Professor A. W. RÜCKER is president of the association and Major P. A. MACMAHON is president of the section of mathematical and physical sciences.

THE Paris academy of sciences proposes the following questions for its mathematical prizes to be awarded at the annual session in December, 1902:—Grand prize of the mathematical section: To perfect in an important point the application of the theory of continuous groups to the study of partial differential equations.—Bordin prize: To develop and perfect the theory of surfaces applicable on the paraboloid of revolution.

THE *Zeitschrift für Mathematik und Physik* (founded by SCHLÖMILCH) was transformed at the beginning of the present year into a journal for applied mathematics only, re-

taining, however, its former name. The editors are to be Professors R. MEHMKE, of Darmstadt, and C. RUNGE, of Hanover. There will also be an editorial committee consisting of Professors Bach, Hauck, Helmert, Klein, Lorentz, Müller, Linde, Seeliger and H. Weber.

UNIVERSITY OF GÖTTINGEN.—The following courses in mathematics are offered during the summer semester, 1901: —By Professor D. HILBERT: Algebra, four hours; Partial differential eq ations (with special regard to their applications), four hours; Seminar on partial differential equations, two hours.—By Professor F. KLEIN: Differential geometry, four hours, with seminar. two hours.—By Professor F. SCHILLING: Analytic geometry, four hours; Graphical statics and exercises.—By Dr. E. ZERMELO: Bessel's functions, two hours; Introduction for physicists, two hours.—By Dr. M. ABRAHAM: Theory of potential, two hours.—By Dr. J. SOMMER: Determinants, one hour; Spherical trigonometry, two hours.

In applied mathematics:—By Professor G. BOHLMANN: Theory of probability, two hours.—By Professor W. VOIGT, Mathematical optics, four hours.—By Professor H. LORENZ, Technical thermodynamics, two hours.

PROFESSOR CHARLES HERMITE, the venerable dean of French mathematicians, died after a brief illness at his home in Paris the 14th day of January, 1901. He was born at Dieuze, December 25th, 1822. He entered the Ecole polytechnique in 1843, for which he was prepared at the lycée Louis-le-Grand. In 1848 he was made répétiteur of analysis and entrance examiner at the Ecole polytechnique; in 1863 he was appointed final examiner, and in 1869, professor of analysis in the same institution, in succession to Duhamel, whom he also succeeded as professor of algebra in the Faculty of sciences of Paris. In 1864 he was appointed maître de conférences at the Ecole normale. In 1856 he was elected to membership in the Paris academy of sciences as the successor of Binet.

MR. H. F. BAKER, F.R.S., has been appointed to a university lectureship in mathematics at Cambridge University.

AT the meeting of the Paris academy of sciences, December 31, 1900, Professor R. DEDEKIND, of the University of Bonn, was elected correspondent of the section of geometry.

MR. S. W. REAVES, graduate scholar in Cornell University, has been appointed instructor of mathematics at Orchard Lake Military Academy.

DR. C. N. LITTLE, professor of mathematics in Leland Stanford University, has resigned his position.

PROFESSOR G. BAUER, of the University of Munich, has been made emeritus professor of mathematics.

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## NEW PUBLICATIONS.

### 1. HIGHER MATHEMATICS.

ASHTON (C. H.). Plane and solid analytic geometry; an elementary text-book. New York, Scribner, 1901. 12mo. 14 + 266 pp. Cloth. $1.00

BORTOLOTTI (E.). Lezioni di calcolo infinitesimale [dettate nell'anno accademico 1899-1900 nella r. università di] Modena. Modena, Pizzolotti, 1899-1900. 8vo. 621 pp.

BÜTTNER (F.). Ein Beitrag zur Theorie der Kugelfunktionen höherer Ordnung. 8vo. 38 pp. (From Festschrift zur 325-jährigen Jubelfeier des fürstl. Stolberg'schen Gymnasiums zu Wernigerode, herausgegeben vom Lehrerkollegium der Anstalt, Leipzig, Fock, 1900.) M. 0.75

CARRÓNE (C.). Le congruenze del secondo ordine senza linee singolari e le loro superficie focali studiate mediante una trasformazione doppia. Catania, 1900. 8vo. 21 pp.

DOBROSERDOV (D. K.). See NERNST (W.).

FORSYTH (A. R.). Theory of functions of a complex variable. 2d edition. London, Clay, 1900. 8vo. 808 pp. 21s.

GAUSS (C. F.). Allgemeine Flächentheorie (Disquisitiones generales circa superficies curvas). (1827.) Deutsch herausgegeben von A. Wangerin. 2te Auflage. Leipzig, Engelmann, 1900. 8vo. 64 pp. Cloth. (Ostwald's Klassiker der exakten Wissenschaften, No. 5.) M. 0.80

GULDBERG (A.). On partial differential equations of the third order. Christiania, Dybwad, 1900. 8vo. 43 pp. (*Videnskabsselskabets Skrifter*, I, Math.-naturv. Klasse, No. 5.)

HALPHEN (G. H.). Traité des fonctions elliptiques et de leurs applications. Partie III: Fragments. (Quelques applications à l'algèbre et particulièrement à l'équation du 5e degré; quelques applications à la théorie des nombres; questions diverses.) Publié par les soins de la Section de Géométrie de l'Académie des Sciences. Paris, Gauthier-Villars, 1901. 8vo. Fr. 8.50

KORN (A.). Lehrbuch der Potentialtheorie. Vol. II: Allgemeine Theorie des logarithmischen Potentials und der Potentialfunktionen in der Ebene. Berlin, Dümmler, 1900. 8vo. 10 + 366 pp. M. 9.00

LIEBMANN (H.). Lehrbuch der Differentialgleichungen. Leipzig, 1901. 8vo. 6 + 226 pp. M. 6.00

LÜROTH (J.). Vorlesungen über numerisches Rechnen. Leipzig, Teubner, 1900. 8vo. 7 + 194 pp. M. 8.00

MILLER (G. A.). On the groups which have the same group of isomorphisms. 1900. 4to. (*Transactions of the American Mathematical Society*, Vol. I, pp. 395–401.)

MOROT. Cours de calcul. Paris, Levé [1900]. 8vo. 79 pp.

NASILIEV (A. V.). See NERNST (W.).

NERNST (W.) and SCHÖNFLIES (A.). Short treatise on the differential and integral calculus for physicists, chemists, and naturalists. Translated into Russian from the second edition by D. K. Dobroserdov and A. V. Nasiliev. Moscow, 1901. 8vo. 15 + 351 pp. (Russian.) M. 6.00

SCHÖNFLIES (A.). See NERNST (W.).

SIMON (M.). Analytische Geometrie der Ebene. 2te Auflage. Leipzig, Göschen, 1900. 12mo. 207 pp. Cloth. (Sammlung Göschen, No. 65.) M. 0.80

WANGERIN (A.). See GAUSS (C. F.).

## II. ELEMENTARY MATHEMATICS.

AHRENS (W.). Mathematische Unterhaltungen und Spiele. (In 2 Hälften.) Leipzig, Teubner, 1901. 8vo. 12 + 428 pp., 1 plate. Cloth. M. 10.00

AUGUST (E. F.). Vollständige logarithmische und trigonometrische Tafeln. 23ste Auflage, besorgt von F. August. Leipzig, 1900. 12mo. 8 + 204 pp. Cloth. M. 1.60

BACKHAUS (K.). See WIESE (B.).

BARBISCH (H.). See JAHNE (J.).

BÉCHÉ (A.). See DUSSAUX (E.).

C. (F. I.). Table de logarithmes à cinq décimales pour les nombres de 1 à 10 000 et pour les fonctions trigonométriques de minute en minute. Edition stéréotype. Paris, Poussielgue [1900]. 16mo. 8 + 148 pp. (Cours de mathématiques élémentaires.)

CRACKNELL (A. G.). Practical mathematics. London, Longmans, 1900. 12mo. 376 pp. (Practical elementary science series.) 3s. 6d.

DUSSAUX (E.) et BÉCHÉ (A.). Première et deuxième années de géométrie dans l'enseignement primaire supérieur. Géométrie plane. Paris, Colin, 1900. 16mo. 344 pp. (Collection Julien Boitel.)

HARTL (H.). Die trigonometrische Auflösung des Dreieckes und der auf Dreiecke zurückzuführenden Figuren, für den Gebrauch an Werkmeister- und Baugewerkeschulen und für den Selbstunterricht bearbeitet. 2te Auflage. Mit 300 Uebungsbeispielen sammt Resultaten, nebst einer Tafel der Winkelfunktionen. Wien, Hölder, 1900. 8vo. 3 + 44 pp. Cloth. M. 0.96

HEGER (R.). Fünfstellige logarithmische, und goniometrische Tafeln, sowie Hülfstafeln zur Auflösung höherer numerischer Gleichungen, für den Gebrauch an höheren Schulen bearbeitet. Leipzig, Teubner, 1900. 8vo. 4 + 112 pp. Cloth. M. 1.60

HENRICI (J.) und TREUTLEIN (P.). Lehrbuch der Elementar-Geometrie. (In 3 Teilen.) Teil III: Die Gebilde des körperlichen Raumes; Abbildung von einer Ebene auf eine zweite (Kegelschnitte). 2te Auflage. Leipzig, Teubner, 1901. 8vo. 12 + 192 pp. Cloth. M. 3.30

HOCHHEIM (A.). Leitfaden für den Unterricht in der Arithmetik und Algebra an höheren Lehranstalten. Heft 1. 6te Auflage, bearbeitet von F. Hochheim. Berlin, Mittler, 1900. 8vo. 6 + 258 pp. M. 3.00

JAHNE (J.) und BARBISCH (H.). Leitfaden der Geometrie und des geometrischen Zeichnens für Bürgerschulen. Ausgabe in 1 Bande. Mit 320 Textfiguren, 156 geometrischen Ornamenten, einer Erklärungstafel für Bezirkspläne und zahlreichen Konstructions- und Rechenaufgaben. Wien, Manz, 1900. 8vo. 8 + 232 pp. Cloth. M. 2.40

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